Notions of local controllability for quantum gates, states and observables

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- There exists a diversity of state spaces that arise in quantum control problems
- Controllability on such state spaces is well-studied (ref dalessandro, rabitz) but not local controllability
- Local controllability is an essential concept for feedback stabilization and control of nonlinear dynamical systems; in classical control logic is at foundation of setpoints - linear quadratic gaussian (LQG) regulator used with local linearization around setpts (ref stengel, b+h)
- Typically start w a canonical optimal control and stabilize system in response to environmental or control noise; expend minimal resources, search effort to return to setpt - how to choose system or control?
- However, the classical definition of local controllability for nonlinear systems, based on control system linearization, is not applicable to quantum control because linearization does not preserve unitarity and probability conservation
- We introduce notions of local controllability suitable to the evolution of bilinear quantum systems on Lie groups; for coherent quantum control the appropriate type of local controllability depends on the state and what quantities are observable

- Two parts: a) for different state spaces and control systems assess rel difficulty of having local controllability; b) for control and stabilization of different types of observables, assess rel difficulty of having local controllability of observable expectation values
- Provide numerical tests

1 Gram, Gramian matrices and local controllability of dynamical systems

A Gram matrix for a sequence of vectors $(v_1, \cdots, v_n)$ is a symmetric positive-definite (spd) matrix of the form

$$G_{ij} = \langle v_i, v_j \rangle, \; i = 1, \cdots, n,$$

that can be used as a test for linear independence of the vectors. Its elements may be written as the inner product of the matrix elements of the outer product of the concatenated vector $w_1 = (v_1, \cdots, v_n)$ in a (real) vector space. From this definition it follows the Gram matrix must be spd. A Gramian matrix is a form of Gram matrix where the vectors $v_i \in L^2$, i.e. functions of time where the inner product $\langle v_i, v_j \rangle = \int_{t_1}^{t_2} v_i(t) v_j(t) \; dt$.

1.1 Local controllability Gramian for nonlinear multivariable control

In nonlinear multivariable control, the *local controllability Gramian* arises explicitly in the formal solution for optimal controls of nonlinear time-variant systems (for Lagrange control problems with specified target final state).

Consider a nonlinear control system $\frac{d}{dt}x(t) = F(x, u, t)$. Denoting the reference trajectory by $x_r(t)$ and the perturbed trajectory by $x(t)$ (from the general solution to a time-variant linear first order ode):

$$x(T) = x_r(T) + U(T)\delta x(0) + \int_0^T U(T,t) B(t) \delta u(t) \; dt \tag{1}$$

where $B(t')$ denotes the $N \times m$ Jacobian matrix (note this assumes a m-dimensional control vector) $\frac{\partial F}{\partial u(t)}$ and $U(T) = \mathrm{T}\exp[\int_0^T \frac{\partial F}{\partial x(t)} \; dt]$ is $N \times N$ (Jacobians evaluated at $x = x_r(t), u = u_r(t)$). Local controllability assesses whether there exists a control perturbation $\delta u(t)$ that can achieve any arbitrary small perturbation at time T from the nominal (reference) trajectory $x_r(T)$. Assume a quadratic Lagrange cost $J(\delta u(t)) = \int_0^T L(\delta u(t))\ dt = -\frac{1}{2}\int_0^T \delta u(t)\ dt$ on the control perturbation, and formulate the control problem of finding a $\delta u(t)$ s.t. $\delta x(T) = dx(T)$; then from the Pontryagin Maximum Principle (PMP, which requires $\frac{\partial \mathbf{H}}{\partial \delta u(t)} = \langle \phi(t), B(t) \rangle = 0$ for optimality, where $\mathbf{H}$ denotes the PMP-Hamiltonian) we have $\delta u(t) = -\langle \phi(t), B(t) \rangle$ with $\phi(t) = \mathbb{T}\exp[\int_t^T A^T(t')\ dt']\phi(T)$. Introduce the $N \times N$ controllability Gramian matrix $G(u,T)$:

$$G(u,T) = \int_0^T U(T,t)B(t)B^T(t)U^T(T,t)\ dt, \tag{2}$$

where $U^T(T,t) = \mathbb{T}\exp[\int_t^T A^T(t')\ dt']$. Then from equation (1) and the expression for $\delta u(t)$,$\phi(T) = G^{-1}(u,T)[dx(T) - U(T)\delta x(0)]$ and the control perturbation $\delta u(t)$ necessary to induce a terminal state increment $dx(T)$ is (assuming $\delta x(0) = 0$)

$$\delta u(t) = B^T(t)U^T(T,t)G^{-1}(u,T)dx(T). \tag{3}$$

A sufficient condition for local controllability is then that the Gramian is nonsingular (ref stengel, syrmos). (The above deriv is std in lit.)

Alternatively (ii), one may obtain the expression for $\delta u(t)$ by solving the Fredholm integral equation (ref int eqn book)

$$\int_0^T U(T,t)B(t)\delta u(t)\ dt = dx(T) \tag{4}$$

for $\delta u(t)$; this involves expanding $\delta u(t)$ on basis $[U(T,t)B(t)]_{ij},\ i = 1,\cdots,N;\ j = 1,\cdots,m$; then $\delta u_i(t) = U(T,t)B(t)c + f(t)$ (where c is a m-dim column vector of expansion coefficients and $f(t)$ is a vector of free functions, since the integral equation is underspecified) and we have

$$\left[\int_0^T U(T,t)B(t)B^T(t)U^T(T,t)\ dt\right] c + \int_0^T f(t)U(T,t)B(t)\ dt = dx(T)$$

Solving for c, we find $c = [\int_0^T U(T,t)B(t)B^T(t)U^T(T,t)\ dt]^{-1}\ [dx(T) - \int_0^T f(t')U(T,t')B(t')\ dt']$. Then $\delta u(t) = B^T(t)U^T(T,t)[\int_0^T U(T,t)B(t)B^T(t)U^T(T,t)\ dt]^{-1}\ [dx(T) - \int_0^T f(t)U(T,t)B(t)\ dt] + f(t)$. The primary difference between the two methods is that the free function in the integral equation approach has a natural interpretation as the partial derivative $\frac{\partial}{\partial u(t)} L(u(t))$ of a Lagrange cost on the field $u(t)$ (which is not of direct interest to local control and stabilization) rather than than its increment $\delta u(t)$.

However, the usefulness of the local controllability Gramian for nonlinear systems depends strongly on the accuracy of the first-order perturbation approximation to the dynamics and hence on the nonlinear order of the control system. Our primary contribution here is to introduce a set of related Gramian matrices for bilinear systems that does not require local linearization of the control system (and hence does not violate unitarity of quantum time evolution).

2 Local controllability of bilinear quantum systems

2.1 Pure state local controllability

As with locally linearized (nonlinear) control systems, the local controllability Gramian matrix for bilinear systems can be derived using either method i) local PMP or ii) integral equation methods. Assume a bilinear quantum control system with a single control $\varepsilon(t)$. (For multiple controls, replace all instances of $\varepsilon(t)$ below with the column vector $\vec{\varepsilon}(t) = (\varepsilon_1(t), \varepsilon_2(t))$.) We start with the problem of pure state local controllability.

i) amounts to the formulation of the Pontryagin maximum principle on the tangent space to the state space. For bilinear systems, instead of locally linearizing the control system, we propagate the original Schrödinger equation with the control, and make the following ansatz for a first order approximation to the evolution of $\delta\psi(T)$ in terms of $\delta\varepsilon(t)$:

$$\begin{aligned}\frac{d\psi(t)}{dt} &= -i(H_0 - \mu \cdot \varepsilon(t))\psi_r(t) + U(t)U^\dagger(T)\frac{\delta\psi(T)}{\delta\varepsilon(t)} \cdot \delta\varepsilon(t) \qquad (5)\\ &= -i(H_0 - \mu \cdot \varepsilon(t))\psi_r(t) + \mu\psi_r(t)\delta\varepsilon(t)\\ &= -i(H_0 - \mu \cdot \varepsilon(t))\psi_r(t) + \mu U(t)\psi(0)\delta\varepsilon(t)\end{aligned}$$

where $\psi(T) \equiv \mathbb{T}\exp[\int_0^T H_0 - \varepsilon(t)\mu \; dt]\psi(0)$; $U(t) \equiv \mathbb{T}\exp[\int_0^t H_0 - \varepsilon(t')\mu \; dt']U(0)$.

This is a linear control system with control now $\delta\varepsilon(t)$ and $B(t) = U(t)U^\dagger(T)\frac{\delta\psi(T)}{\delta\varepsilon(t)}$; now apply the formal linear solution: $\psi_r(T) + \delta\psi(T) = U(T)\psi_r(0) + U(T)\int_0^T U^\dagger(t)U(t)U^\dagger(T)\frac{\delta\psi(T)}{\delta\varepsilon(t)}\ dt$, which verifies our ansatz (5).

Assume as above that the objective is to locally minimize a quadratic Lagrange cost on the control, i.e., (analogous to equation (3)) $J(\delta\varepsilon(\cdot)) = -\frac{1}{2}\int_0^T \delta\varepsilon(t)\ dt$, while inducing a terminal state increment $d\psi(T)$. We solve $\max\limits_{\delta\varepsilon(\cdot)} J$ by applying the Pontryagin maximum principle (ref jurdjevic):

$$\frac{\partial \mathbf{H}}{\partial\delta\varepsilon(t)} = -\delta\varepsilon(t) + \langle\phi(t), U(t)U^\dagger(T)\frac{\delta\psi(T)}{\delta\varepsilon(t)}\rangle = 0$$
$$\delta\varepsilon(t) = \langle\phi(t), U(t)U^\dagger(T)\frac{\delta\psi(T)}{\delta\varepsilon(t)}\rangle$$

where $\phi(t) \in T_{\psi(t)}S_{\mathcal{H}_\mathcal{N}}$ and $\frac{\delta\psi(T)}{\delta\varepsilon(t)} = U(T)U^\dagger(t)\psi(0)$. Since $\phi(t) = U(t,T)\phi(T)$ (backwards propagation of the costate following the Schrödinger equation),

$$\begin{aligned}\delta\varepsilon(t) &= \langle U(t)U^\dagger(T)\phi(T), U(t)U^\dagger(T)\frac{\delta\psi(T)}{\delta\varepsilon(t)}\rangle \\ &= \langle\phi(T), \frac{\delta\psi(T)}{\delta\varepsilon(t)}\rangle.\end{aligned}$$

Thus

$$\psi_r(T) + d\psi(T) = U(T)\psi_r(0) + \left[\int_0^T \frac{\delta\psi(T)}{\delta\varepsilon(t)}\frac{\delta\psi^\dagger(T)}{\delta\varepsilon(t)}\ dt\right]\phi(T)$$
$$d\psi(T) = \left[\int_0^T \frac{\delta\psi(T)}{\delta\varepsilon(t)}\frac{\delta\psi^\dagger(T)}{\delta\varepsilon(t)}\ dt\right]\phi(T) \tag{6}$$

so

$$\phi(T) = \left[\int_0^T \frac{\delta\psi(T)}{\delta\varepsilon(t)}\frac{\delta\psi^\dagger(T)}{\delta\varepsilon(t)}\ dt\right]^{-1} d\psi(T)$$

and

$$\delta\varepsilon(t) = \frac{\delta\psi^\dagger(T)}{\delta\varepsilon(t)}\left[\int_0^T \frac{\delta\psi(T)}{\delta\varepsilon(t)}\frac{\delta\psi^\dagger(t)}{\delta\varepsilon(t)}\ dt\right]^{-1} d\psi(T). \tag{7}$$

The local controllability Gramian is thus

$$G_\psi(\varepsilon, T) = \int_0^T \frac{\delta\psi(T)}{\delta\varepsilon(t)}\frac{\delta\psi^\dagger(T)}{\delta\varepsilon(t)}\ dt. \tag{8}$$

To generalize the above result to m control inputs, arrange $U(T)U^\dagger(t)\mu_1\psi(t), U(T)U^\dagger(t)\mu_2\psi(t)$ vectors in m columns of a matrix that plays a role analogous to $B(t)$ above.

$\frac{\delta\psi(T)}{\delta\varepsilon(t)}$ should lie in the tangent space to the complex sphere, $\mathcal{T}_\psi S_\mathcal{H}$ or equivalently the tangent space to the complex projective space $\mathcal{T}_\psi \mathbb{CP}^{N-1}$. This is a direct product of tangent space to torus and tangent space to quadrant of hypersphere; **simple to parameterize but harder to get coordinate-independent rep; do last** It may be shown that $U(T)U^\dagger(t)\mu\psi(t)$ has components orthogonal to this tangent space. An equivalent Gramian, expressed in terms of a Hilbert sphere tangent vector $\frac{\delta\psi(T)}{\delta\varepsilon(t)}$, is

$$
\begin{aligned}
G_{\psi'}(\varepsilon, T) &= \int_0^T \nu[\psi(T)\psi^\dagger(t)\mu\psi(t)][\psi(T)\psi^\dagger(t)\mu\psi(t)]dt \\
&= \int_0^T \nu[iU(T)\psi(0)\psi^\dagger(0)U^\dagger(t)\mu U(t)\psi(0)]\nu^T[iU(T)\psi(0)\psi^\dagger(0)U^\dagger(t)\mu U(t)\psi(0)]\ dt.
\end{aligned}
$$

Local controllability requires the Gramian is of rank $2N-1$ (this includes global phase control). This is a sufficient condition for local controllability if $\frac{\delta\psi^\dagger(t)}{\delta\varepsilon(t)}$ is represented on a $2N-1$ dimensional basis. An example of a minimal parameterization of $S_\mathcal{H}$ is the Hurwitz parameterization (ref Hurwitz, book), $\psi(\theta,\phi) = (r_1(\theta)\exp(i\phi_1), \cdots, r_N(\theta)\exp(i\phi_N))$; the parameters are amplitudes and complex phases. In the Hurwitz parameterization,

$$
\nu[\psi(T)\psi^\dagger(t)\mu\psi(t)] = ([\psi(T)\psi^\dagger(t)\mu\psi(t)]_{\theta_1}, \cdots, [\psi(T)\psi^\dagger(t)\mu\psi(t)]_{\theta_{N-1}}, \psi(T)\psi^\dagger(t)\mu\psi(t)]_{\phi_1}, \cdots, \psi(T)\psi^\dagger(t)\mu\psi(t)]_{\phi_N}).
$$

If a minimal parametrization of the pure state space is not used, rank $G \geq 2N-2$ is a necessary but not sufficient condition for local controllability, since it does not guarantee that *any* $dU(T)\psi(0)$ can be reached by appropriate choice of $\delta\varepsilon(t)$.

Numerical methods for the computation of such Gramians are presented below.

The main difference with respect to nonlinear local controllability (2) is the use of $U(t)U^\dagger(T)\frac{\delta\psi(T)}{\delta\varepsilon(t)}$ in place of the Jacobian $\frac{\partial F}{\partial\varepsilon(t)}$ for the matrix $B(t)$ in the locally linearized control system. This provides an exact condition for whether any first order variation $d\psi(T)$ can be reached by a control perturbation $\delta\varepsilon(t)$. Note that due to the bilinearity of the control system, the fundamental matrix $A(t) = i(H_0 - \varepsilon(t)\mu)$ is identical with or without control system linearization. $\varepsilon(t)$ for which the Gramian matrix is singular are often referred to as singular controls (ref wu).

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2.2 Local operator controllability

An analogous method can be applied to derive the $\delta\varepsilon(t)$ that induces a prespecified first order variation $dU(T)$ in the unitary propagator rather than a pure state. Here,

$$
\begin{aligned}
\delta\varepsilon(t) &= \langle \phi(t), U_r(t)U_r^\dagger(T)\frac{\delta U(T)}{\delta\varepsilon(t)}\rangle \\
&= \langle \phi(t), U_r(t)U_r^\dagger(T)\frac{\delta U(T)}{\delta\varepsilon(t)}\rangle \\
&= \langle \phi(T), \frac{\delta U(T)}{\delta\varepsilon(t)}\rangle \\
&= \mathrm{Tr}\left[\phi^\dagger(T), \frac{\delta U(T)}{\delta\varepsilon(t)}\right] \\
&= \ \nu^T\left[\frac{\delta U^\dagger(T)}{\delta\varepsilon(t)}\right]\nu[\phi(T)]
\end{aligned}
$$

since $\phi(t) = U_r(t)U_r^\dagger(T)\phi(T)$. Here we have used the notation $U_r(t)$ to explicitly denote the reference trajectory of the propagator; henceforth the subscript will be omitted. So from the formal solution to the corresponding linear ode

$$
\nu[dU(T)] = \int_0^T \nu\left[\frac{\delta U(T)}{\delta\varepsilon(t)}\right]\nu^T\left[\frac{\delta U^\dagger(T)}{\delta\varepsilon(t)}\right]\ dt\ \phi(T).
$$

Solving for the control perturbations $\delta\varepsilon(t)$, we obtain the equivalent expressions

$$
\delta\varepsilon(t) = \nu^T\left[\frac{\delta U(T)}{\delta\varepsilon(t)}\right]\left\{\int_0^T \nu[U(T)\mu(t)]\nu^T[\mu(t)U^\dagger(T)]dt\right\}^{-1}\nu[dU(T)] \tag{9}
$$

and

$$
\begin{aligned}
\delta\varepsilon(t) &= \nu^T\left[\frac{\delta A(T,\varepsilon)}{\delta\varepsilon(t)}\right]\left\{\int_0^T \nu[\mu(t)]\nu^T[\mu(t)]dt\right\}^{-1}\nu[dA(T)] \\
&= \nu^T[\mu(t)]\left\{\int_0^T \nu[\mu(t)]\nu^T[\mu(t)]dt\right\}^{-1}\nu[dA(T)]
\end{aligned} \tag{10}
$$

where $U^\dagger(T)dA(T,\varepsilon) = dU(T)$.

Reachability of any $dU(T)$ ($dA(T)$) by appropriate choice of $\delta\varepsilon(\cdot)$ thus requires that the $N^2 \times N^2$ Gramian matrix

$$
G_U(\varepsilon, T) = \int_0^T \nu[U(T)\mu(t)]\nu^T[\mu(t)U^\dagger(T)]dt \tag{11}
$$

(12)

(where $v[\cdot]$ represents the vectorization of an $N \times N$ complex matrix into a N^2-length complex vector), is full rank. or equivalently the nonsingularity of

$$S_U(\epsilon, T) = \int_0^T \nu[\mu(t)]\nu[\mu(t)]^T \, dt, \tag{13}$$

i.e., that the independent real and imaginary components of the matrix elements of $\mu(t)$ are linearly independent functions of time. [1]

The expected condition number of this matrix, i.e., the ratio of the largest to the smallest singular values (or the measure of singular Gramians on $\mathbb{K}$), is larger than that for G, since local controllability on $S_{\mathcal{H}}$ imposes fewer constraints on $\varepsilon(t)$ than local controllability on $\mathcal{U}(N)$.

For control systems with multiple inputs, S_U becomes

$$S_U(\vec{\varepsilon}, T) = \int_0^T (\nu[\mu_1(t)], \nu[\mu_2(t)], \cdots)(\nu[\mu_1(t)], \nu[\mu_2(t)], \cdots)^T \, dt.$$

where $(\nu[\mu_1(t)], \nu[\mu_2(t)], \cdots)$ is a matrix with columns $\nu[\mu_i(t)]$, analogous to $B(t)$ above. (10) indicates that local controllability for such bilinear systems is equivalent to reachability on the Euclidean Lie algebra rather than the Lie group, which results in important differences in terms of the necessary conditions for local controllability and controllability of bilinear systems; the relationship is discussed further below.

Alternatively, integral equation methods can be used to obtain these Gramians. For bilinear systems, the integral equation can be written in two equivalent forms (compare eqn (4) above):

$$dU(T) = \int_0^T \frac{\delta U(T)}{\delta \varepsilon(t)} \delta\varepsilon(t) \, dt \tag{14}$$

where $dU(T) \equiv U(T, \varepsilon + \delta\varepsilon) - U(T, \varepsilon)$, or

$$\int_0^T U^\dagger(T) \frac{\delta U(T)}{\delta \varepsilon(t)} \delta\varepsilon(t) \, dt = dA(T). \tag{15}$$

To solve Fredholm integral equation (14), one expands $\delta\varepsilon(t)$ on basis functions that are the independent parameters of $U(T)\mu(t)$, whereas for integral equation (14), the expansion is carried out on the independent parameters of $\mu(t)$ (ref pareto, multiobs).

[1] We note that analogously to $S_U(\epsilon, T)$, there is a Gramian $S_\psi(\epsilon, T)$ corresponding to G_ψ for pure states wherein $\frac{\delta\psi(T)}{\delta\varepsilon(t)} \mapsto U^\dagger(T) \frac{\delta\psi(T)}{\delta\varepsilon(t)}$ (the tangent space at the identity operator rather than $U(T)$.

2.3 Mixed state local controllability

We have shown that assessment of local pure state controllability based on Gramian rank requires minimal parameterizations of the state space while local operator controllability does not (the latter only requires retention of unitary time evolution). This circumstance extends to general density matrices. Consider first the density matrix representation of a pure state.

For any problem where $\rho_0 = U(T)|\psi(0)\rangle\langle\psi(0)|U^\dagger(T)$, $U(T)$ may be parameterized according to the coset $\frac{U(N)}{U(N-1)} \approx \mathbb{CP}^{N-1}$, since $U(N-1)$ is the stabilizer in $SU(N)$; then, $\psi(t)$ is a unitary eigenvector. The local control problem may then be formulated on the coset rather than on the Hilbert sphere/complex projective space. Full rank of the associated Gramian would then be equivalent to local controllability on the homogeneous space $\frac{U(N)}{U(N-1)}$.

More generally we may consider local controllability of mixed states. Since we are concerned with coherent local controllability, the state space must be restricted to unitarily equivalent density matrices, i.e., the orbit $\mathcal{O}_{\mathcal{U}(N)}[\rho_0] = \mathcal{O}_{\mathcal{U}(N)}[D]$, where ρ_0 denotes the initial density matrix and D denotes the diagonal matrix of eigenvalues of ρ. A "minimal" parametrization of $\mathcal{O}_{\mathcal{U}(N)}[\rho_0]$ is of the form given above for pure states but with $U(T) \in U(N)/stab(D)$, the latter being a flag manifold/homogeneous space. Thus, $U(N)/stab(D) = U(N)/U(1)^{SU(2)} \times \cdots \times U(1)^{SU(N)} = U(N)/T^N = SU(N)/T^{N-1}$, where $T^{N-1} = U(1) \times \cdots \times U(1)$ denotes the maximal torus subgroup. In the coset parameterization of ρ, the density matrix is parameterized by $N^2 - N$ eigenvector parameters in the unitary coset (for a fully mixed state) and $N-1$ eigenvalue parameters; the $N-1$ eigenvalue parameters of the matrix are unchanged by coherent control. We next derive the form of the local controllability Gramian matrix for minimally parameterized density matrices.

For the linearized system analogous to (5), we have

$$\frac{d\rho(t)}{dt} = -i[H_0 - \mu \cdot \varepsilon(t), \rho(t)] + U^\dagger(t)U(T)\frac{\delta\rho(T)}{\delta\varepsilon(t)}U^\dagger(T)U(t) \cdot \delta\varepsilon(t)$$

with solution

$$\rho(T) + \delta\rho(T) = U(T)\rho_0 U^\dagger(T) + \int_0^T \frac{\delta\rho(T)}{\delta\varepsilon(t)}\, dt.$$

The PMP now demands $\frac{\partial \mathbf{H}}{\partial \delta\varepsilon(t)} = -\delta\varepsilon(t) + \langle \phi(t), U(t)U^\dagger(T)\frac{\delta\rho(T)}{\delta\varepsilon(t)}U(T)U^\dagger(t)\rangle = 0$. The matrix costate

$\phi(t)$ is now an element of the tangent space to $\mathcal{O}_{\mathcal{U}(N)}[\rho_0]$, which is

$$\mathcal{T}_{\rho(T)}\mathcal{O}_{\mathcal{U}(N)}[\rho_0] = \{U(T)AU^\dagger(T)|\ A^\dagger = -A,\ A \in \frac{u(N)}{\mathcal{C}[\rho_0]}\}.$$

Here $A \in \frac{u(N)}{\mathcal{C}[\rho_0]}$, the orthogonal complement (quotient set) of the centralizer of ρ_0 in $u(N)$ (set of all skew-Hermitian matrices that do not commute with ρ_0).

To write $\phi(t)$ in terms of $\phi(T)$, we determine $\phi(t)$ s.t.

$$\langle \phi(t), -i[\mu, U(t)\rho_0 U^\dagger(t)]\rangle = i\mathrm{Tr}\{\phi^\dagger(T)U(T)[U^\dagger(t)\mu U(t), \rho_0]\};$$

$$\phi^\dagger(t) = U(t)U^\dagger(T)\phi^\dagger(T)U(T)U^\dagger(t),$$

which constitutes backwards propagation of $\phi^\dagger(T)$ as a Hermitian operator following the vN equation $\frac{d\phi(t)}{dt} = -i[H(t), \phi(t)]$. Since $\phi(t) = U(t)U^\dagger(T)\phi(T)U(T)U^\dagger(t)$,

$$\begin{aligned}\delta\varepsilon(t) &= \langle U^\dagger(t)U(T)\phi(T), U(t)U^\dagger(T)\frac{\delta\rho(T)}{\delta\varepsilon(t)}U(T)U^\dagger(t)\rangle \\ &= \langle \phi(T), \frac{\delta\rho(T)}{\delta\varepsilon(t)}\rangle.\end{aligned}$$

Note that the following tangent space is simpler to parameterize than $\mathcal{T}_{\rho(T)}\mathcal{O}_{\mathcal{U}(N)}[\rho_0]$:

$$\mathcal{T}_{U(T)}\frac{\mathcal{U}(N)}{\mathrm{stab}[\rho_0]} = \{U(T)A|\ A^\dagger = -A,\ A \in \frac{u(N)}{\mathcal{C}[\rho_0]}\};$$

Formulating the local control problem with $\phi(T) \in \mathcal{T}_{U(T)}\frac{\mathcal{U}(N)}{\mathrm{stab}[\rho_0]}$, we get

$$\delta\varepsilon(t) = \langle \phi(T), \frac{\delta U(T)}{\delta\varepsilon(t)}\rangle$$

where $U(T) \in \frac{U(N)}{\mathrm{stab}[\rho_0]}$, whereas with $\phi(T) \in \frac{u(N)}{\mathcal{C}[\rho_0]}$ we have

$$\delta\varepsilon(t) = \langle \phi(T), U^\dagger(T)\frac{\delta U(T)}{\delta\varepsilon(t)}\rangle. \tag{16}$$

For the Gramian we need a parameterization of $\frac{u(N)}{\mathcal{C}[\rho_0]}$ (to represent the functions $[\mu(t), \rho_0]$). From equation (16), analogously to the derivation for operator local controllability,

$$\begin{aligned}\nu[dA(T)] &= \left\{\int_0^T \nu\{[\mu(t),\rho_0]\}\nu^T\{[\mu(t),\rho_0]\}\ dt\right\}^{-1}\phi(T) \\ &:= S_\rho(\varepsilon, T)\phi(T). \end{aligned}\tag{17}$$

Since $\phi(T) = S_\rho^{-1}\,\nu[dA(T)]$, we have $\delta\varepsilon(t) = \nu^T\{[\mu(t),\rho_0]\}\left\{\int_0^T \nu\{[\mu(t),\rho_0]\}\nu^T\{[\mu(t),\rho_0]\}dt\right\}^{-1}\,\nu[dA(T)]$ for minimal fluence local control.

Local mixed state controllability (global phase-insensitive) can be assessed using the Gramian matrix in (17), or either

$$G_\rho(\varepsilon,T) = \int_0^T \nu\{U(T)[\mu(t),\rho_0]U^\dagger(T)\}\nu^T\{U^\dagger(T)[\mu(t),\rho_0]U(T)\}\ dt \tag{18}$$

or

$$G_{U,\rho}(\varepsilon,T) = \int_0^T \nu\{U(T)[\mu(t),\rho_0]\}\nu^T\{[\mu(t),\rho_0]U^\dagger(T)\}\ dt. \tag{19}$$

Again it is convenient to express the Gramian in terms of reachability (local controllability) on the Lie subalgebra (i.e., using (**??**)) rather than the unitary coset. An advantage of the coset parameterization of quantum states as far as local controllability is concerned is that this (bijective) mapping of the state increment to a Euclidean space is possible - this simplifies analysis of the effect of control system Hamiltonian on local controllability, as will be shown below in Section 4.

The above Gramians can be used for *all* observable control problems (Section 3) since the state space for observable control is always a submanifold of the Bloch vector space of density matrices. For fully nondegenerate mixed states, the Gramian matrix is of order $N^2 - N$ (the dimension of the coset $\frac{U(N)}{T^N}$). In general, the Gramian matrix is of order $\dim(\frac{u(N)}{\mathcal{C}[\rho_0]}) = \dim(\frac{U(N)}{\mathrm{stab}_{U(N)}[\rho_0]}) = N^2 - \dim(U(m_1)\times\cdots\times U(m_k)) = N^2 - \sum_i m_i^2$, where m_i denotes the degeneracy of eigenvalues i of ρ and $U(m_1)\times\cdots\times U(m_k)$ hence represents the stabilizer of ρ in $\mathcal{U}(N)$.

Numerically, for any ρ_0, the Gramian can be represented in the Euler coset parameterization (ref tilma) as follows. Any $A \in \frac{u(N)}{\mathcal{C}[\rho_0]}$ can be vectorized in the EP as

$$\nu(A) = \left(c_1 U^\dagger \frac{\partial U(\alpha)}{\partial\alpha_1}, c_2 U^\dagger \frac{\partial U(\alpha)}{\partial\alpha_2}, \cdots, c_n U^\dagger \frac{\partial U(\alpha)}{\partial\alpha_n}\right)\Big|_{\alpha=0},$$

where $U(\alpha)$ parameterizes $\frac{U(N)}{stab[\rho_0]}$ and $c_i = \mathrm{Tr}[AU^\dagger \frac{\partial U(\alpha)}{\partial\alpha_i}\ |_{\alpha=0}]$; then $\nu(A) = (c_1,\cdots,c_n)$. Thus we can write

$$\nu\left[\frac{\delta A(T)}{\delta\varepsilon(t)}\right] = \left(\mathrm{Tr}\left[\frac{\delta A(T)}{\delta\varepsilon(t)}U^\dagger\frac{\partial U(\alpha)}{\partial\alpha_1}\right],\ \cdots\ ,\mathrm{Tr}\left[\frac{\delta A(T)}{\delta\varepsilon(t)}U^\dagger\frac{\partial U(\alpha)}{\partial\alpha_n}\right]\right)\Big|_{\alpha=0}.$$

For example, for $N = 2$ pure state local controllability, we have $U(\alpha) = \exp(i\lambda_3\alpha_1)\exp(i\lambda_2\alpha_2)$ for the Euler parameterization of $\frac{U(2)}{U(1)}$ (ref Tilma) and $\nu[A] = (\mathrm{Tr}[iA\lambda_2], \mathrm{Tr}[iA\,(\lambda_2 \sin(\alpha_1) + \lambda_3 \cos(\alpha_1))])$ for $A \in \frac{u(2)}{\mathcal{C}[|\psi_0\rangle\langle\psi_0|]}$, where the λ_i are the generators of $SU(2)$. Thus

$$[\mu(t), |\psi(0)\rangle\langle\psi(0)|] = i\nu\{[\mu(t), |\psi(0)\rangle\langle\psi(0)|]\}_1\lambda_2 + i\nu\{[\mu(t), |\psi(0)\rangle\langle\psi(0)|]\}_2(\lambda_2 \sin(\alpha_1) + \lambda_3 \cos(\alpha_1)). \tag{20}$$

(include derivation here or ref raj)

In order to compute the local state controllability Gramian numerically:

1. Propagate $\rho(t)$ using the von Neumann equation.

2. Expand $[\mu, \rho(t)]$ on basis found above for $\frac{u(2)}{\mathcal{C}[\rho_0]}$

3. Use e.g. (20) together with (17) to obtain the Gramian.

3 Local controllability of quantum observable expectation values: pure and mixed states

1. This requires nonzero gradient; necc cond is that $\frac{dU_T(s)}{ds} \neq 0$ (or approp analog); only study this cond

2. Applications beyond common goal of observable maximization - any problem where observable expectation value must be controlled/stabilized

For locally uncontrollable quantum systems, certain outputs may still be locally controllable. A common quantum control goal is the achievement and stabilization of the expectation value of a Hermitian observable Θ. In optimal control theory, this objective can be naturally framed in terms of a so-called Mayer cost functional, with $F(U(T)) = \mathrm{Tr}(U(T)\rho U^\dagger(T)\Theta)$. According to the PMP for Mayer functionals (ref B+H), the terminal costate $\phi(T) = \nabla_x F(x(T))$. From equation (17), then,

$$dU(T) = \nu^{-1}\left\{G_{U,\rho}(\varepsilon, T)\nu[\nabla F(U(T))]\right\}$$

or equivalently

$$dU(T) = U(T)\nu^{-1}\left\{S_{U,\rho}(\varepsilon,T)\nu[U^\dagger(T)\nabla F(U(T))]\right\} \tag{21}$$

where $U(T) \in \frac{U(N)}{\text{stab}_{U(N)}[\rho_0]}$. This expression can be used to derive a necessary and sufficient condition for the existence of a control perturbation $\delta\varepsilon(t)$ that, to first order, can correct for any state perturbation $dU(T)$ that alters the setpoint observable expectation value - or, that can induce an arbitrary small change $d\langle|\Theta(T)|\rangle$ in the setpoint:

$$U^\dagger(T)dU(T) = \nu^{-1}\left\{S_{U,\rho}(\varepsilon,T)\nu[U(T)^\dagger\nabla F(U(T))]\right\} = 0. \tag{22}$$

The solution set to this homogeneous system of algebraic equations is a subset of $\frac{u(N)}{\mathcal{C}(\rho_0)}$. (To solve for the kernel, replace $U^\dagger(T)\nabla F(U(T))$ in (22) with $\phi(T) \in \frac{u(N)}{\mathcal{C}(\rho_0)}$.)

The Gramians are matrix representations of linear maps that operate on the costate vector $\phi(T)$. For example, for Gramians (17,18, 19), the associated linear mappings are (using the same notation for the map and matrix)

1. $S_{U,\rho}(\varepsilon,T):\ \frac{u(N)}{\mathcal{C}[\rho_0]} \to \frac{u(N)}{\mathcal{C}[\rho_0]}$.
2. $G_\rho(\varepsilon,T):\ \mathcal{T}_{U(T)}\mathcal{O}_{U(N)}[\rho_0] \to \mathcal{T}_{U(T)}\mathcal{O}_{U(N)}[\rho_0]$
3. $G_{U,\rho}(\varepsilon,T):\ \mathcal{T}_{U(T)}\frac{U(N)}{\text{stab}[\rho_0]} \to \mathcal{T}_{U(T)}\frac{U(N)}{\text{stab}[\rho_0]}$,

respectively.

Let $\{v_1,\cdots,v_m\}$ denote the eigenvectors spanning the range of the symmetric matrix S, i.e., range $G =$ span$\{v_1,\cdots,v_m\}$ where span denotes the linear span of vectors . Appending these column vectors provides the order N^2 orthogonal matrix X such that $S_{U,\rho_0}(\varepsilon,T) = X\tilde{S}_{U,\rho_0}(\varepsilon,T)X^T$ where $\tilde{S}_{U,\rho_0}(\varepsilon,T) = \text{diag}(s_1,\cdots,s_m,\underbrace{0,\cdots,0}_{N^2-m})$. (Written in terms of the Gramian $G_{U,\rho}(\varepsilon,T)$, we have $G_{U,\rho}(\varepsilon,T) = X\tilde{G}_{U,\rho}(\varepsilon,T)Y^T$, where X denotes the matrix of left singular vectors of G, and $\tilde{G}$ is a real, positive semidefinite diagonal matrix.) In this basis equation (21) becomes $dU(T) = U(T)\nu^{-1}\left\{X\tilde{S}\tilde{\nu}[U(T)^\dagger\nabla F(U(T))]\right\}$. The eigenvectors of $G_{U,\rho}(\varepsilon,T)$ are $N^2 - \sum_i m_i$ orthogonal directions in the tangent space to $\frac{\text{stab}U(N)}{U(\mathbf{m})}$ at $U(T)$, and the corresponding eigenvalues are a measure of each direction's contribution to the variation $\delta U(T)$.

The norm of the orthogonal projection of ∇F onto the nullspace of G can be written

$$\sqrt{\sum_i |\langle \nabla F(U), v_i \rangle|^2} \tag{23}$$

(analogously for projection of $U^\dagger \nabla F(U)$ onto the nullspace of S). This norm is zero if $\nabla F(U)$ lies in the nullspace of G. The norm increases with increasing colinearity of $\nabla F(U)$ and the eigenvectors of G corresponding to larger eigenvalues.

The eigenvalue spectrum of the observable operator Θ plays a central role in determining local expectation value controllability, according to (21). Let $\tilde{U}(T) \equiv W^\dagger U(T) V$, where W is the matrix of eigenvectors of Θ and V is the matrix of eigenvectors of ρ, and denote by $U(\mathbf{n}) =\equiv U(n_1) \times \cdots U(n_l)$ the stabilizer of the diagonalized observable operator $\tilde{\Theta}$. In general the submanifold of $\frac{U(N)}{U(\mathrm{m}_1 \times \cdots \times U(m_k)}$ that produces equivalent values of $F(U)$ (level set of F) is of dimension

$$\dim \frac{U(N)}{U(\mathbf{m})} - 1 \geq \dim \mathcal{M} \geq \dim U(\mathbf{m}) \cap U(\mathbf{n}), \tag{24}$$

since any element of the intersection of the stabilizers preserves both ρ and Θ, but this is not a necessary condition to reside on the level set. The precise dimension of the level set changes with the value of F - since $F(\tilde{U}(T)) = \sum_{i,j} |\tilde{U}_{ij}(T)|^2 \lambda_i \gamma_j$, i.e. a weighted sum of the eigenvalues λ_i, γ_j of ρ_0, Θ respectively, certain values of F are associated with a higher dimensional subset of $\tilde{U}_{ij}(T)$ due to degenerate matchings of eigenvalues. However, in general the level set dimension increases with eigenvalue degeneracy (symmetries) in Θ. Note that the symmetries in Θ reduce the dimensionality of the solution set to equation (22) below that of the state manifold coset $\frac{\mathcal{U}(N)}{\mathcal{U}(\mathbf{m})}$ (whose dimension is set solely by the eigenvalue symmetries of ρ). These symmetries reduce the number of independent variables in $\nabla F(U)$ because the gradient is orthogonal to the level set, which results in constraints on $\phi(T)$ ([2]). Since these constraints render it less likely that elements of $\ker(S)$ have nonzero projections onto the nullspace of S, symmetries in Θ generally increase the expected norm of dU.

Unlike state local controllability, assessment of observable expectation value local controllability does not require a minimal parameterization. For observable expectation value control and stabilization, the

[2] However, we have not proven here that $\phi(T)$ lies in a proper subalgebra of $\frac{u(N)}{\mathcal{C}[\rho_0]}$.

set of time-evolved states $\rho(T) = U(T)\rho_0 U^\dagger(T)$ such that ∇F lies in the nullspace of the Gramian is equivalent irrespective of whether a representation on $u(N)$ or $\frac{u(N)}{\mathcal{C}[\rho_0]}$ is used; this is a result of the additional symmetries of $\nabla F(U(T))$ in the former case. However, note again that whereas rank $G_{U,\rho}(\varepsilon, T) = N^2 - N$ implies rank $G_U(\varepsilon, T) \geq N^2 - N$ (for a fully nondegenerate mixed state), the converse is not necessarily true.

It may be shown (ref multiobs, kateraj) that equation (21) specifies the change in the unitary propagator per step of a steepest ascent control optimization algorithm that aims to maximize the expectation value of Θ (those for other algorithms may be determined through appropriate choice of the costate ϕ). If its eigenvalues are all far from zero then $S(\epsilon, T)$ is *well-conditioned* and it follows that during the course of control optimization by first-order algorithms, the gradient norm can only grow infinitesimally small at the global optimum of the control landscape (ref kateraj).

4 Controllability vs local controllability: necessary conditions for local controllability

Here we formulate necessary conditions for local state controllability of a quantum control system based on its drift and control Hamiltonians. The effect of control system Hamiltonians on rank of the associated Gramian matrix is also studied. We show that global state controllability is not a sufficient condition for local controllability.

4.1 State controllability and equality of orbits

Prior work (dalessandro) has examined sufficient conditions for full state controllability, which can be expressed as equality of orbits:

$$\mathcal{O}(\rho_0) = \{U\rho_0 U^\dagger | U \in \exp(\mathcal{L})\} = \{U\rho_0 U^\dagger | U \in \exp(u(N))\}. \tag{25}$$

(This is a necessary but not sufficient condition for a full rank Gramian matrix, i.e. rank $S_{U,\rho}(\varepsilon, T) =$ dim $\frac{U(N)}{U(\mathbf{m})}$. In this Section we provide more stringent necessary conditions for local state controllability.

4.2 Magnus expansion and local state controllability

In the Schrödinger picture, the Magnus series can provide a necessary condition for local state controllability in terms of the commutators of the drift H_0 and control μ Hamiltonians. The Magnus series represents the controlled propagator as $U(T) = \exp(A(T, \varepsilon(t))$, with (no ref, derive in sep paper)

$$
\begin{aligned}
A(T,\varepsilon(t)) = H_0 - \mu\varepsilon(t) - \frac{1}{2!}[H_0,\mu]\int_0^t\int_0^{t'} \varepsilon(t'') - \varepsilon(t')\ dt''dt - \frac{1}{12}[H_0,[H_0,\mu]]\int_0^t\int_0^{t'}\int_0^{t''} \varepsilon(t''') - \varepsilon(t'')\ dt'''dt''dt' + \\
-\frac{1}{4}[H_0,[H_0,\mu]]\int_0^t\int_0^{t'}\int_0^{t'} \varepsilon(t''') - \varepsilon(t'')\ dt'''dt''dt' + \\
+\frac{1}{12}[\mu,[H_0,\mu]]\int_0^t \varepsilon(t')\int_0^{t'}\int_0^{t''} \varepsilon(t''') - \varepsilon(t'')\ dt'''dt''dt' + \\
+\frac{1}{4}[\mu,[H_0,\mu]]\int_0^t \varepsilon(t')\int_0^{t'}\int_0^{t'} \varepsilon(t''') - \varepsilon(t'')\ dt'''dt''dt' + \cdots
\end{aligned}
$$

In this representation we may write

$$
\begin{aligned}
&U(T,\varepsilon(t)) + \delta U(T,\delta\varepsilon(t)) = U(T,\varepsilon(t))\exp[A(T,\delta\varepsilon(t))] \\
&\frac{\delta U(T,\varepsilon(t))}{\delta\varepsilon(t)} = \frac{\delta}{\delta\varepsilon(t)}\left\{U_T(\varepsilon(t))\exp\left[H_0 - \mu\delta\varepsilon(t) - \frac{1}{2!}[H_0,\mu]\int_0^T\int_0^{t'} \delta\varepsilon(t'') - \delta\varepsilon(t')\ dt''dt' - \cdots\right]\right\}\Big|_{\delta\varepsilon(\cdot)=0} \\
&= U(T,\varepsilon(t))\frac{\delta}{\delta\varepsilon(t)}\Big\{\mu - \frac{1}{2!}[H_0,\mu]\int_0^t dt - \frac{1}{2!}[H_0,\mu]\int_0^T\int_0^{t'} T\delta\varepsilon(t')dt' - \frac{1}{4}[\mu,[H_0,\mu]]\Big(\int_0^T \varepsilon(t')\int_0^{t'}\int_0^{t''} \delta\varepsilon(t''' \\
&\quad - \delta\varepsilon(t'')\ dt'''dt''dt' + \int_0^T\int_0^{t'}\int_0^{t''} \varepsilon(t''') - \varepsilon(t'')\ dt'''dt''dt'\Big) - \cdots\Big\} \\
&= U(T)\left\{\mu - \frac{1}{2!}[H_0,\mu](T-t) - \frac{1}{4}[H_0,[H_0,\mu]](\frac{T^2}{2!} - Tt) + \frac{1}{4}[\mu,[H_0,\mu]]\int_0^T \varepsilon(t')\int_0^t dt'dt + \cdots\right\}
\end{aligned}
$$

Let $\frac{\delta}{\delta\varepsilon(t)}A(T,\delta\varepsilon(t))$ denote the term in curly brackets. Provided that the series converges, the local state controllability Gramian in (17) could be expressed $\int_0^T \nu[\frac{\delta}{\delta\varepsilon(t)}A(T,\delta\varepsilon(t)),\rho_0]\nu^T[\frac{\delta}{\delta\varepsilon(t)}A(T,\delta\varepsilon(t)),\rho_0]\ dt$ and we would have $\frac{\delta}{\delta\varepsilon(t)}A(T,\delta\varepsilon(t)) = \mu(t)$. However, the Magnus expansion has a finite radius of convergence, and the above expression is derived under the assumption that the expansion for $U(T,\varepsilon(t))$ converges. Hence it is necessary to subdivide the domain of integration into successive intervals, on each of which the expansion for $U(T,\varepsilon(t))$ is guaranteed to converge (guaranteed if $\int_{t_1}^{t_2} H(t)\ dt \leq \pi$, ref magnus convergence). Thus we write

$$
\begin{aligned}
U(T) &= U(T,t_{n-1})\cdots U(t_2,t_1)\\
U(T)+\delta U(T) &= U(T,t_{n-1})\exp(A(T,t_{n-1}))U(t_{n-1},t_{n-2})\cdots U(t_2,t_1)+\cdots+U(T,t_{n-1})\cdots U(t_2,t_1)\exp(A(t_2,t_1))\\
\frac{\delta U(T)}{\delta\varepsilon(t)} &= U(T,t_{n-1})\exp(A(T,t_{n-1}))U(t_{n-1},t_{n-2})\cdots U(t_2,t_1)+\cdots+U(T,t_{n-1})\cdots U(t_2,t_1)\exp(A(t_2,t_1))\big|_{\varepsilon(\cdot)=0}\\
&= \{U(T,t_{n-1})A(T,t_{n-1})U(t_{n-1},t_{n-2})\cdots U(t_2,t_1)+\cdots+U(T,t_{n-1})\cdots U(t_2,t_1)A(t_2,t_1)\}\,\big|_{\varepsilon(\cdot)=0}
\end{aligned}
$$

Note this requires us to keep matrix exponentials and hence does not allow analytic characterization of the necessary conditions for local controllability. Instead, we consider local controllability on each interval $[t_i,t_{i+1}]$, which if satisfied for all i, implies local controllability on $[0,T]$ since $\mathcal{U}(N)$ is a compact Lie group that can be finitely generated (ref dalessandro). (This is analogous to the classic treatment of local controllability of time-variant nonlinear systems, without the approximation of system linearization.) Thus consider

$$U(t_{i+1},t_i)+\delta U(t_{i+1},t_i)=U(t_{i+1},t_i)\exp(A(t_{i+1},t_i))$$

Operate under the assumption of a uniformly bounded field ($|\varepsilon(t)|\le c,\ t\in[t_i,t_{i+1}],\ \forall i$); hence $\varepsilon(t),\ t\in[t_i,t_{i+1}]$ is bounded by the same constant for all i. Then, the necessary local controllability condition becomes identical on each interval and hence only one condition need be checked.

Analytically, a necessary condition for (unitarily equivalent) local state controllability is:

$$\text{rank}\left\{\underbrace{[[H_0,\mu],\rho_0],[[H_0,[H_0,\mu]],\rho_0],[[\mu,[H_0,\mu]],\rho_0],\cdots}_{k}\right\}=\dim\left(\frac{U(N)}{U(m_1)\times\cdots\times U(m_n)}\right). \quad (26)$$

where the number of commutators in brackets k depends on the uniform field bound c. (An analogous expression for the number of commutators of H_0,μ required for global state controllability does not exist because equality of orbits for global controllability requires matrix exponentiation of the series expansion of commutators.) Singular value decomposition of the matrix formed by adjoining the vectorized commutators provides a basis for a dynamical subalgebra of $\frac{u(N)}{\mathcal{C}[\rho_0]}$. Determination of the integer k corresponding to a given uniform field bound and error residual for $||A_k(T,\delta\varepsilon(t)) - A_k(T,\delta\varepsilon(t))||$ requires a convergence analysis of the Magnus expansion. In general, control systems which require more commutators (hence higher uniform bound c) to satisfy (26) will have more ill-conditioned Gramian matrices. If the number of commutators required for field bound c is known, a corresponding approximate Gramian on $\frac{u(N)}{\mathcal{C}[\rho_0]}$ for local controllability on the interval $[t_1, t_2]$ can be computed under the Magnus expansion as

$$S_{U,\rho_0}(\varepsilon,T) \approx \int_{t_1}^{t_2} \nu \left\{ [\mu,\rho_0] - \frac{1}{2!}[[H_0,\mu],\rho_0](T-t) - \frac{1}{4}[[H_0,[H_0,\mu]],\rho_0](\frac{T^2}{2!} - Tt) + \cdots \right\}$$
$$\nu^T \left\{ [\mu,\rho_0] - \frac{1}{2!}[[H_0,\mu],\rho_0](T-t) - \frac{1}{4}[[H_0,[H_0,\mu]],\rho_0](\frac{T^2}{2!} - Tt) + \cdots \right\} dt.$$

to compare the local state controllability of quantum control systems based on their H_0, μ. This will be pursued in a separate work.