# Series expansions for propagator first-order variation: necessary conditions for local gate controllability

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This note studies the Dyson series expansion for landscape first-order variation, applying the Cayley-Hamilton theorem in order to obtain a necessary condition for local controllability given the uniformly bounded controls and control variations that are typically encountered over the course of control field optimization. The condition is suggested to be an indicator for whether dynamical properties of the control system may slow control optimization; it can be checked for various control systems of interest in quantum computation. If the condition is satisfied, convergence rate may be only weakly influenced by dynamical properties of the control system. To the extent that diverse quantum systems may satisfy the condition at low field strengths, it may help characterize the "scale similarity" of control optimization search effort.

## 1 Necessary condition for local operator controllability: Dyson series representation

Let $H_0$ denote the skew-symmetric matrices obtained by multiplying the field-free Hamiltonian by $\imath$. The controlled unitary propagator (in the interaction picture) can be expressed in terms of the Dyson series expansion (ref):

$$
\begin{aligned}
U_I(T) &= \mathrm{T}\exp\left[-\frac{i}{\hbar}\int_0^T H_I(t)\ dt\right] \\
&= I_N - \frac{i}{\hbar}\int_0^T H_I(t)\ dt + \left(-\frac{i}{\hbar}\right)^2\int_0^T H_I(t^1)\int_0^{t^1} H_I(t^2)\ dt^2dt^1 + \left(-\frac{i}{\hbar}\right)^3\int_0^T H_I(t^1)\int_0^{t^1} H_I(t^2)\int_0^{t^2} H_I(t^3)\ dt^3dt^2 \\
&= I_N + \frac{i}{\hbar}\int_0^T V^\dagger(t^1)\mu V(t^1)\varepsilon(t^1)\ dt^1 + \left(\frac{i}{\hbar}\right)^2\int_0^T V^\dagger(t^1)\mu V(t^1)\varepsilon(t^1)\int_0^{t^1} V^\dagger(t^2)\mu V(t^2)\varepsilon(t^2)\ dt^2dt^1 + \cdots \\
&= I_N + \frac{i}{\hbar}\sum_{i,j=0}^{N-1} H_0^i\mu H_0^j\int_0^T a_i(t^1)a_j(t^1)\varepsilon(t^1)\ dt^1 + \\
&\left(-\frac{i}{\hbar}\right)^2\sum_{i,j,k,l=0}^{N-1} H_0^i\mu H_0^j H_0^k\mu H_0^l\int_0^T a_i(t^2)a_j(t^2)\varepsilon(t^2)\int_0^{t^2} a_k(t^3)a_l(t^3)\varepsilon(t^3)\ dt^2dt^1 + \cdots \qquad (1)
\end{aligned}
$$

where $V(t) = \exp(-iH_0t)$; $U(t) = V(t)U_I(t)V^\dagger(t)$, and where in the last line we have applied the Cayley-Hamilton theorem, which states that the matrix exponential $V(t)$ can be expressed as a matrix polynomial of (at most) order $N$: $V(t) = a_0(\lambda,t)I_N + a_1(\lambda,t)(iH_0) + \cdots + a_{N-1}(\lambda,t)(iH_0)^{N-1}$ where $\lambda$ denotes the vector of eigenvalues of $(i)H_0$. (This follows because any matrix $A^n$ in the matrix Taylor series for $\exp(A)$ can be written as a linear combination of $A^i,\ 1 \le i \le N-1$, due to the fact that the matrix $A$ satisfies its own characteristic polynomial. Note that $\sum_{i,j=0}^{N-1} a_i(t)H_0^i$ is unitary, and $\sum_{i,j=0}^{N-1} a_i(t)a_j(t)H_0^i\mu H_0^j$ is Hermitian, although the individual terms in these sums need not be. For skew-symmetric matrices $A$, $\imath a_i \in \mathbb{R}$.) We henceforth use units $\hbar = 1$.

The convergence of series (1) can be proven for fields $\varepsilon(t)$ of arbitrary power, as follows. Let $|\langle x|\sum_{i,j=0}^{N-1} H_0^i\mu H_0^j\varepsilon(t)|y\rangle| < c,\ 1 \le x,y \le N$. $(|\langle x|\varepsilon(t)V(t)\mu V^\dagger(t)|y\rangle| < c)$, for some positive constant $c$. Then

$$
\begin{aligned}
&\left|\sum_{x_1=1}^N\cdots\sum_{x_{k-1}=1}^N\int_0^T\langle x|\varepsilon(t)V^\dagger(t)\mu V(t)|x_1\rangle\cdots\int_0^{t^{k-1}}\langle x_{k-1}|\varepsilon(t^k)V^\dagger(t^k)\mu V(t^k)|y\rangle\ dt^k\cdots dt\right| \\
&\le \sum_{x_1=1}^N\cdots\sum_{x_{k-1}=1}^N\int_0^T|\langle x|\varepsilon(t)V^\dagger(t)\mu V(t)|x_1\rangle|\cdots\int_0^{t^{k-1}}|\langle x_{k-1}|\varepsilon(t^k)V^\dagger(t^k)\mu V(t^k)|y\rangle|\ dt^k\cdots dt \\
&\le \sum_{x_1=1}^N\cdots\sum_{x_{k-1}=1}^N c^k\int_0^T\cdots\int_0^{t^{k-1}} dt^kdt \\
&\le N^{k-1}\frac{c^kT^k}{k!} = \frac{(NcT)^k}{Nk!}
\end{aligned}
$$

Thus

$$|\langle x|U_I(T)|y\rangle| \leq \sum_{k=0}^{\infty} \frac{N^k T^k c^k}{Nk!} + 1$$
$$\leq \frac{1}{N}\exp(NTc) + 1, \tag{2}$$

and for any uniform bound on the field intensity, we have a bound on the magnitude of the elements of the interaction picture propagator. Since the series is absolutely bounded from above, it converges. For a specified convergence tolerance $d$, there exists an integer $n$ such that the series in the interaction picture converges at order $n$ - i.e., denoting by $\epsilon_n \equiv |\langle x|U_I(T) - U_I^n(T)|y\rangle|$ the maximum error residual of the series (1) for all $x, y$ at order $n$, $\epsilon_n \leq d$. Denote by $U_I^n(\varepsilon(\cdot), T)$ the series (1) truncated at order $n$.

Unlike the Lie algebra rank condition for bilinear system controllability, which applies for unconstrained controls, a controllability condition based on the Dyson series expansion (1) can be formulated for uniformly bounded controls, which is useful for our purposes. Define d-*approximate operator controllability* as the existence, for each $W \in U(N)$, of a control $\varepsilon(t)$ such that $|\langle x|U(\varepsilon, T) - W|y\rangle| \leq d$. Now consider the $2N^2 \times N^{2n}$ matrix $F_c^d$ whose rows consist of the real and imaginary parts of the elements of $H_0^i \mu H_0^j, H_0^i \mu H_0^j H_0^k \mu H_0^l, \cdots$ and whose columns are indexed by all combinations $0 \leq i, j, k, l, \cdots \leq N-1$ up to order $n$, according to equation (1). We prove

**Theorem 1** *A necessary condition for the d-approximate operator controllability of the quantum system* with uniform bound $c$ on $|\langle x|\sum_{i,j=0}^{N-1} H_0^i \mu H_0^j a_i(t) a_j(t)|y\rangle|\varepsilon(t), \quad x, y = 1, \cdots, N$ *is* rank $F_c^d \geq N^2$*, i.e., that* $F_c^d$ *has at least* $N^2$ *linearly independent rows (or equivalently, columns).*

**Proof.** Let $\mathbb{K} = L^2[0,T]$ denote the space of control fields on $[0,T]$, and denote by $\nu(X)$ the vectorization of a $N \times N$ complex matrix $X$ into a $2N^2$-dimensional real vector. Operator controllability is equivalent to the requirement that the map $U(\varepsilon, T) : \mathbb{K} \to \mathcal{U}(N)$ is onto. Assume that for each of a linearly independent set of $N^2 + 1$ linearly independent vectors $v \in \mathbb{R}^{2N^2}$, the inner product $\langle v, \sum_{i,j=0}^{N-1} \nu(H_0^i \mu H_0^j) u_{ij} + \sum_{i,j,k,l=0}^{N-1} \nu(H_0^i \mu H_0^j H_0^k \mu H_0^l) u_{ijkl} + \cdots \rangle = 0$ for all $u_{ij}, u_{ijkl}, \cdots \in \mathbb{R}$. If such a set of $v$'s exists, rank $F < N^2$. Then $\langle v, \nu(H_0^i \mu H_0^j \mu \cdots)\rangle = 0, \; 0 \leq i, j, \cdots \leq N-1$ for each $v$. It follows that for each $v$, $\langle v, \nu(H_0^i \mu H_0^j \mu \cdots)\varepsilon(t)\rangle = 0$ for all $\varepsilon(t)$ satisfying the above bound. The converse is not true[1], since $\langle v, U_I^n(\varepsilon(\cdot),T)\rangle = 0$ for all $\varepsilon(\cdot)$ does not imply $\langle v, \nu(H_0^i\mu H_0^j\mu\cdots)\rangle$ for all $i,j,k,l,\cdots$, since all integrals in the series (1) for $U_I^n(\varepsilon(\cdot),T)$ need not be independent. ■

In practice, for any nonzero $d$, it is possible that rank $F_c^d > N^2$, since $U_I^n$ need not be unitary. However, our primary interest is in comparing the eigenvalue spectra of $F_c^d$ for various control systems.

The first order variation in the interaction picture propagator due to a variation $\delta\varepsilon(t)$ in the control is

$$\begin{aligned}\delta U_I(T) = i\sum_{i,j=0}^{N-1} H_0^i\mu H_0^j \int_0^T a_i(t)a_j(t)\delta e(t^1)\,dt^1 + \\ -\sum_{i,j,k,l=0}^{N-1} H_0^i\mu H_0^j H_0^k\mu H_0^l\Bigg[\int_0^T a_i(t^1)a_j(t^1)\delta\varepsilon(t^1)\int_0^{t^1} a_k(t^2)a_l(t^2)\varepsilon(t^2)\,dt^2dt^1 + \\ \int_0^T a_i(t^1)a_j(t^1)e(t^1)\int_0^{t^1} a_k(t^2)a_l(t^2)\delta\varepsilon(t^2)\,dt^2dt^1\Bigg] + \cdots \end{aligned} \tag{3}$$

so the functional Jacobian, which appears in the landscape gradient eqn (6), is

$$\begin{aligned}\frac{\delta U_I(T)}{\delta\varepsilon(t)} &= \sum_{i,j=0}^{N-1} H_0^i\mu H_0^j a_i(t^1)a_j(t^1) + \\ &+ \sum_{i,j,k,l=0}^{N-1} H_0^i\mu H_0^j H_0^k\mu H_0^l\left[a_k(t^2)a_l(t^2)\int_0^T a_i(t^1)a_j(t^1)\varepsilon(t^1)dt^1 + a_i(t^1)a_j(t^1)\int_0^{t^1} a_k(t^2)a_l(t^2)\varepsilon(t^2)\,dt^2\right] + \cdots \\ &= \sum_{i,j=0}^{N-1} H_0^i\mu H_0^j a_i(t)a_j(t) + \\ &\sum_{i,j,k,l=0}^{N-1} H_0^i\mu H_0^j H_0^k\mu H_0^l\left[a_k(t)a_l(t)\int_0^T a_i(t^1)a_j(t^1)\varepsilon(t^1)\,dt^1 + a_i(t)a_j(t)\int_0^t a_k(t^1)a_l(t^1)\varepsilon(t^1)\,dt^1\right] + \cdots \end{aligned} \tag{4}$$

because $t^1, t^2, t^3, \cdots$ are dummy variables. This entails the dynamical contribution to the landscape gradient equation (6). Let $\epsilon_n \equiv |\langle x|\delta U_I(T) - \delta U_I^n(T)|y\rangle|$ denote the error residual of the series (3) for element $x, y$ at order $n$. To derive a bound on the error residual for the series expansion for the first

[1]The analogous Kalman controllablity rank condition is sufficient for linear systems.

variation at order $n$, consider the norms of the elements $\langle x|\delta U_I(T)|y\rangle$ of the first variation:

$$|\langle x|\sum_{i,j=0}^{N-1} H_0^i\mu H_0^j \int_0^T a_i(t)a_j(t)\delta e(t^1)\ dt^1 + \sum_{i,j,k,l=0}^{N-1} H_0^i\mu H_0^j H_0^k \mu H_0^l \Big[\int_0^T a_i(t^1)a_j(t^1)\delta\varepsilon(t^1)\int_0^{t^1} a_k(t^2)a_l(t^2)\varepsilon(t^2)\ dt^2 dt^+ \cdots$$

$$+\int_0^T a_i(t^1)a_j(t^1)e(t^1)\int_0^{t^1} a_k(t^2)a_l(t^2)\delta\varepsilon(t^2)\ dt^2 dt^1\Big] + \cdots |y\rangle|$$

$$\leq \sum_{i=1}^{\infty}\frac{bT^i(Nc)^{i-1}}{(i-1)!} = bT\exp(NTc). \qquad (5)$$

where $b \geq |\langle i|\sum_{i=0}^{N-1} H_0^i\mu H_0^j a_i(t)a_j(t)\delta\varepsilon(t)|j\rangle|,\ \forall\ 0\leq t\leq T$. Thus we have

**Lemma 1** *If the upper bound on the error residual $\epsilon_n$ for the propagator series (1), computed from equation (2) is d, the corresponding upper bound for the error residual $\epsilon_n$ for series (3) is* **...***, since if* $|\frac{1}{N}\exp(NTc) - \sum_{i=0}^{n}\frac{(NTc)^i}{Ni!}| \leq d$*, then* $|bT\exp(NTc) - \sum_{i=1}^{n}\frac{bN^iT^ic^{i-1}}{(i-1)!}| \leq$ .....

Define *d-approximate local controllability* as the existence, for each $\delta W \in \mathcal{T}_W U(N)$, of a control variation $\delta\varepsilon(t)$ such that $|\langle x|\delta U(\varepsilon, T) - \delta W|y\rangle| \leq d$. Let $F_{b,c}^d$ be defined analogously to $F_c^d$ above. We now prove that

**Theorem 2** *A necessary condition for the d-approximate local controllability (nonsingularity of the Gramian matrix $G_\varepsilon(T)$) of the quantum system* with uniform bound $b$ on $|\langle x|\sum_{i,j=0}^{N-1} H_0^i\mu H_0^j a_i(t)a_j(t)|y\rangle|\delta e(t)$ *and uniform bound $c$ on $|\langle x|\sum_{i,j=0}^{N-1} H_0^i\mu H_0^j a_i(t)a_j(t)|y\rangle|\varepsilon(t)$ is that* rank $F_{b,c}^d \geq N^2$*, i.e., that $F_{b,c}^d$ has $N^2$ linearly independent rows (or equivalently, columns).*

**Proof.** -Local controllability- is equivalent to the requirement that the map $dV_T : \delta\varepsilon(\cdot) \to \mathcal{T}_U\mathcal{U}(N)$ is onto. Assume that for each of a set of $N^2+1$ linearly independent vectors $v \in \mathbb{R}^{2N^2}$, the inner product $\langle v, \sum_{i,j=1}^{N}\nu(H_0^i\mu H_0^j)u_{ij} + \sum_{i,j,k,l=1}^{N}\nu(H_0^i\mu H_0^j H_0^k\mu H_0^l)u_{ijkl} + \cdots\rangle = 0$ for all $u_{ij}, u_{ijkl}, \cdots \in \mathbb{R}$. If such a set of $v$'s exists, rank $F < N^2$. Then $\langle v, \nu(H_0^i\mu H_0^j\cdots)\rangle = 0,\ 0 \leq i,j,\cdots \leq N-1$. It follows that $\langle v, \sum_{i,j,\cdots=0}^{N-1}\nu(H_0^i\mu H_0^j\cdots)\delta\varepsilon(t)\rangle = 0$, for all $\delta\varepsilon(t)$ satisfying the bound above. The converse is not true for bilinear systems, since $\langle v, \delta U_I^n(\delta\varepsilon(\cdot), T)\rangle = 0$ for all $\delta\varepsilon(\cdot)$ does not imply $\langle v, \nu(H_0^i\mu H_0^j)\rangle = 0$ for all $i,j$. Hence rank $F_c^d < N^2$ implies rank $G_\varepsilon(T) < N^2$ but not vice versa. ■

Direct computation of rank F may be numerically inconvenient since the dimensions of $F$ scale steeply with $n$, and in the absence of mechanistic information, $n$ must be computed according to the bound on field amplitude. An analysis that surmounts these limitations, based on the CBH expansion for the controlled propagator, is in progress. However, since $F$ is time-independent, unlike $G(T)$, and only depends on the control through the bounds in eqn (8) and Lemma 1, it allows analytic assessment of necessary conditions for local controllability [2].

The condition number of the time-invariant matrix $F_{b,c}^{d}$ can significantly affect the convergence rate of optimization algorithms, since a high condition number implies that generation of certain directions in $\mathcal{T}_U\mathcal{U}(N)$ (which may be necessary to attain the target transformation $W$) requires higher field intensities and/or particular field modes in $\delta\varepsilon(t)$, which may not be locally accessible by the trajectory (7) of first-order optimization algorithms. Because of the relation in Lemma 1, the Dyson series for the first-order variation (3) converges at roughly the same order as the series (1) for $U_I(T)$. $b$ is determined by the uniform bound (8) on $\delta\varepsilon(t)$ in first-order gradient-based optimization algorithms (7). Thus in gradient flow algorithms, where $\delta\varepsilon(s,t) = \frac{\delta J(\varepsilon_s)}{\delta\varepsilon(t)}$, search trajectories with ill-conditioned controllability matrices $F_s$ may only slowly add field modes contained in the functions $\Re\langle i|\mu(t)|j\rangle$, $\Im\langle i|\mu(t)|j\rangle$ to $\varepsilon_s(t)$.

Thus we conclude that the rate at which the rate at which rank $F$ is saturated with respect to Dyson series order $n$, or **equivalently** the rate at which the dynamical Lie algebra is saturated by the nested commutators $[H_{i_1}, \cdots, [H_{i_{k-1}}, H_{i_k}]]$, plays an important role – in addition to the trap-free control landscape topology – in explaining the observed rapid convergence to optimal gate controls for certain Hamiltonians. Subsequent work will explore these properties for diverse quantum systems of interest for computation.

The initial field guess $\varepsilon_0(t)$ for quantum control optimization algorithms is often parameterized in terms of frequency modes corresponding to direct (single photon) transitions. However, direct transitions are typically insufficient to achieve an arbitrary unitary transformation (or even arbitrary state-state transition) in a prescribed time. Full controllability of the quantum system, which is required to reach

[2]The Gramian matrix $G$ is the analog of the local controllability Gramian that arises classical control engineering of time variant (non)linear systems. $F$ is the bilinear analog of the Kalman controllability matrix, which provides sufficient conditions for controllability for time-invariant linear systems. $F$ introduced above provides necessary, but not sufficient conditions for local controllability of bilinear systems with constrained controls.

any $W \in \mathcal{U}(N)$, typically requires the introduction of Lie algebra directions corresponding to higher order commutators $[H_{i_1}, \cdots, [H_{i_{k-1}}, H_{i_k}]]$ (where $H_{i_j}$ denotes one of the set $\{H_0, \mu\}$), which only enter the Dyson series in higher order terms (ref). Higher order terms in the Dyson expansion are required in the optimal control mechanism (ref) for control systems with greater dynamical Lie algebra depth, which corresponds to higher $n$ and column rank in $F_c^d$. Moreover, the amplitudes of these field modes must increase with the depth of the commutator to which they correspond (**order at which the product** $H_0^i \mu H_0^j$ enters the Dyson series). In principle, field modes corresponding to higher order terms in the Dyson series could be incorporated into $\varepsilon_0(t)$, but the Dyson integrals are expensive to compute numerically. Generally, field modes contributing to higher order series terms must be incorporated into the control over the course of the optimization trajectory (7).

Restrictions on the power spectrum of $\delta\varepsilon(t)$ can also exclude higher series orders from contributing to $\delta U_I(T)$ within a given convergence tolerance, according to quantum control mechanism analysis (hr check). For any given $\delta\varepsilon(t)$, it is possible to determine with mechanism analysis which modes contribute dominantly to the series expansion (ref). The gradient $\frac{\delta J}{\delta\varepsilon(t)}$ at any step of control optimization can only introduce field modes contained within functions $\Re\langle i|\mu(t)|j\rangle$, $\Im\langle i|\mu(t)|j\rangle$ in fixed relative amplitudes. Assuming that the field modes contributing to higher order Dyson series terms do not have large amplitudes in $\frac{\delta J}{\delta\varepsilon(t)}$ at the outset of optimization, their amplitudes must be progressively increased over the optimization trajectory. The Hessian (curvature) determines the rate at which modes can be added to $\delta\varepsilon(t)$ in new relative proportions, with magnitude of the eigenvalues of the Hessian being proportional to the rates at which the new modes contained within the eigenfunctions of $\mathcal{H}(t, t')$ are added.

## 2 Necessary condition for local operator controllability: Magnus series representation

In the Magnus series representation, $U_T(\varepsilon(t)) = \exp(A_T(\varepsilon(t))$, where

$$
\begin{aligned}
A_T(\varepsilon(t)) = H_0 - \mu\varepsilon(t) - \frac{1}{2!}[H_0,\mu]\int_0^T\int_0^{t'} \varepsilon(t'') - \varepsilon(t')\ dt''dt' - \frac{1}{12}[H_0,[H_0,\mu]]\int_0^T\int_0^{t'}\int_0^{t''} \varepsilon(t''') - \varepsilon(t'')\ dt'''dt''dt' + \\
- \frac{1}{4}[H_0,[H_0,\mu]]\int_0^T\int_0^{t'}\int_0^{t''} \varepsilon(t''') - \varepsilon(t'')\ dt'''dt''dt' + \\
+ \frac{1}{12}[\mu,[H_0,\mu]]\int_0^T \varepsilon(t')\int_0^{t'}\int_0^{t''} \varepsilon(t''') - \varepsilon(t'')\ dt'''dt''dt' + \\
+ \frac{1}{4}[\mu,[H_0,\mu]]\int_0^T \varepsilon(t')\int_0^{t'}\int_0^{t''} \varepsilon(t''') - \varepsilon(t'')\ dt'''dt''dt' + \cdots .
\end{aligned}
$$

In this representation we may write

$$
\begin{aligned}
U_T(\varepsilon(t)) + \delta U_T(\delta\varepsilon(t)) &= U_T(\varepsilon(t))\exp[A_T(\delta\varepsilon(t))] \\
\frac{\delta U_T}{\delta\varepsilon(t)} &= \frac{\delta}{\delta\varepsilon(t)}\left\{U_T(\varepsilon(t))\exp\left[H_0 - \mu\delta\varepsilon(t) - \frac{1}{2!}[H_0,\mu]\int_0^T\int_0^{t'} \delta\varepsilon(t'') - \delta\varepsilon(t')\ dt''dt' - \cdots\right]\right\}|_{\delta\varepsilon(\cdot)=0} \\
&= U_T(\varepsilon(t))\left\{\mu - \frac{1}{2!}[H_0,\mu]\int_0^t dt - \frac{1}{2!}[H_0,\mu]\int_0^T\int_0^{t'} T\delta\varepsilon(t')dt' - \frac{1}{4}[\mu,[H_0,\mu]]\left(\int_0^T \varepsilon(t')\int_0^{t'}\int_0^{t''} \delta\right.\right. \\
&= U(T)\left\{\mu - \frac{1}{2!}[H_0,\mu](T-t) - \frac{1}{4}[H_0,[H_0,\mu]](\frac{T^2}{2!} - Tt) + \cdots\right\}
\end{aligned}
$$

So in the Magnus expansion

$$
\mu(t) = U^\dagger(T)\frac{\delta U_T}{\delta\varepsilon(t)} = \mu - \frac{1}{2!}[H_0,\mu](T-t) - \frac{1}{4}[H_0,[H_0,\mu]](\frac{T^2}{2!} - Tt) + \cdots .
$$

However, the Magnus expansion has a finite radius of convergence, and the above expression is derived under the assumption that the expansion for $U_T(\varepsilon(t))$ converges. Hence it is necessary to subdivide the domain of integration into intervals, on each of which the expansion for $U_T(\varepsilon(t))$ is guaranteed to converge. Thus we write

$$
\begin{aligned}
U(T) &= U(T,t_{n-1})\cdots U(t_2,t_1) \\
U(T)+\delta U(T) &= U(T,t_{n-1})\exp(A(T,t_{n-1}))U(t_{n-1},t_{n-2})\cdots U(t_2,t_1)+\cdots+U(T,t_{n-1})\cdots U(t_2,t_1)\exp(A(t_2,t_1)) \\
\frac{\delta U(T)}{\delta\varepsilon(t)} &= U(T,t_{n-1})\exp(A(T,t_{n-1}))U(t_{n-1},t_{n-2})\cdots U(t_2,t_1)+\cdots+U(T,t_{n-1})\cdots U(t_2,t_1)\exp(A(t_2,t_1))_{\varepsilon(\cdot)=0} \\
&= \{U(T,t_{n-1})A(T,t_{n-1})U(t_{n-1},t_{n-2})\cdots U(t_2,t_1)+\cdots+U(T,t_{n-1})\cdots U(t_2,t_1)A(t_2,t_1)\}_{\varepsilon(\cdot)=0}
\end{aligned}
$$

Note this requires us to keep matrix exponentials and hence does not allow analytic characterization of the necessary conditions for local controllability. Instead, we consider local controllability on each interval $[t_i,t_{i+1}]$, which if satisfied for all $i$, implies local controllability on $[0,T]$ since $\mathcal{U}(N)$ is a compact Lie group that be finitely generated. (This is analogous to the classic treatment of local controllability of time-variant nonlinear systems, without the approximation of linearization.) Thus consider

$$U(t_{i+1},t_i)+\delta U(t_{i+1},t_i) = U(t_{i+1},t_i)\exp(A(t_{i+1},t_i))$$

From the above analysis, we have as a necessary and sufficient condition for local operator controllability

$$\int_0^T \nu([\mu(t),\rho(0)])\nu^T([\mu(t),\rho(0)])\ dt$$

is full rank, which implies that the map $d_\varepsilon\rho_T:\ \mathbb{K}\to\to T_{\rho(T)}\mathcal{O}_{\mathcal{U}(N)}[\rho(0)]$ is surjective. Note this is not an analytic condition. Analytically, a necessary condition is (need to do convergence analysis for truncation of Magnus series, probably in interaction pict):

$$\operatorname{rank}\{[[H_0,\mu],\rho(0)],[[H_0,[H_0,\mu]],\rho(0)],[[\mu,[H_0,\mu]],\rho(0)],\cdots\} = \dim\left(\frac{U(N)}{U(m_1)\times\cdots\times U(m_n)}\right\}.$$

Comparing the necessary and sufficient condition for global state controllability, we note that the local controllability condition with bounded controls does not involve all elements of the Lie algebra generated by $H_0,\mu$. Operate under assumption of uniformly bounded field; hence $\varepsilon(t),\ t\in[t_i,t_{i+1}]$ is bounded by the same constant for all $i$. Then, the necessary local controllability condition becomes identical on each interval and hence only one condition need be checked.

# A Landscape gradient and bounds on first-order variation

$$\frac{\delta J}{\delta \varepsilon(t)} = -i\mathrm{Tr}\left(\left[W^\dagger U(T,0) - U^\dagger(T,0)W\right] U^\dagger(t,0)\mu U(t,0)\right) = 0. \tag{6}$$

$$\frac{\partial \varepsilon(s,t)}{\partial s} = \alpha(s)\ \frac{\delta J}{\delta \varepsilon(s,t)}, \tag{7}$$

$$|\nabla J(\varepsilon(t))| \leq \sqrt{T} 2N ||\mu||, \tag{8}$$